On the Theory of Randomly Misaligned Beam Waveguides

FELIX K. SCHWERING, MEMBER, IEEE

Abstract—A straightforward method for the determination of the expected attenuation in misaligned beam waveguides is presented. It applies to confocal guides and assumes that the misalignment consists of random displacements of the lenses in directions perpendicular to the beam axis.

Reiterative fields, as they are present in perfectly aligned beam waveguides, do not exist in misaligned guides. However, it can be shown that there are beams whose expected field distribution is repeated from lens to lens. These "statistical modes" are determined by the eigenfunctions of a homogeneous integral equation of the second kind. The corresponding eigenvalues determine the expected attenuation per iteration. The absolute squares of the eigenvalues yield an upper bound for the expected power loss per iteration.

The integral equation is solved for small mean square displacements of the lenses by a perturbation method. For infiitely extended lenses, the equation can be solved in closed form. In both cases the expected attenuation of the lower-order statistical modes has been calculated; the results are shown as function of the mean square displacement of the lenses.

Introduction

THE MODE THEORY of beam waveguides [1]–[3] is derived with the assumption that the guiding structure is strictly periodic. Apart from other conditions, this requires that the phase transforming lenses be perfectly aligned on the waveguide axis. A transmitted field can then be described by a system of reiterative wavebeams (so-called beam modes) whose cross-sectional field distribution is repeated with the spacing of the lenses. With every iteration, the beam modes are multiplied by a complex amplitude factor determining the attenuation and phase shift per iteration. The purpose of this paper is to establish the effects of random misalignments of the lenses on the field distribution and the attenuation. The study is limited to so-called confocal guides where the spacing of the lenses is twice their focal length. Furthermore, the assumption is made that the lenses are displaced only in a direction perpendicular to the waveguide axis. Displacements in the direction of the waveguide axis and tilting of the lenses leads to smaller-order effects and will, therefore, not be considered here.[1] The lateral displacement of each lens is assumed to have a Gaussian probability distribution, with a standard displacement which is the same for all the lenses.

We will show that in this type of misaligned guiding structure modes exist that are reiterative in the statistical sense; this means the cross-sectional distribution of the expected fields is repeated from lens to lens. Similar to the case of perfect alignment, the statistical modes are determined by the eigenfunctions of a homogeneous integral equation of the second kind. The eigenvalues of this integral equation determine the expected attenuation between two successive lenses.

Misaligned beam waveguides have been treated in the literature from two points of view. A number of papers [4]–[8] deal with the path of the beam axis and the variation of the beam diameter in a misaligned beam waveguide, but do not include the effect of misalignment on attenuation and transmission loss. Eaglesfield [9] was the first to investigate the expected loss of the dominant mode in a misaligned waveguide. He expands any transmitted wavebeam into the mode functions of the corresponding aligned guide with infinite apertures; the misalignment of the lenses then leads to an interaction between the different mode functions so that a wavebeam passing through a misaligned guide is described by a spectrum of modes which in general changes from lens section to lens section. Eaglesfield assumes that, after the wavebeam has passed a large number of lenses, a quadratic steady state will be reached, i.e., a steady state characterized by an expected power spectrum of the modes that does not change any longer apart from a factor common to all modes that determines the expected power loss per iteration. This expected power loss has been calculated approximately for the dominant mode, assuming small mean square displacements of the lenses. Gloge [10] has studied wavebeam resonators with tilted mirrors, and from his analysis derives the field distribution and attenuation in systematically misaligned beam waveguides, such as periodically misaligned or continuously curved waveguides. He also treats randomly misaligned waveguides. The approximation used in this case, however, considers only the loss caused by a conversion of a mode, incident on a displaced lens, into a mode spectrum, but neglects the partial reconversion of this spectrum into the initial mode by the following displaced lenses.

The integral equation derived in this paper formulates (within the framework of the Fresnel-Kirchhoff theory) the misalignment problem rigorously, and yields the field distribution of the expected mode functions and the expected amplitude attenuation per iteration, though not the expected power loss. In the limiting case of infinitely extended lenses

Manuscript received August 29, 1966; revised November 28, 1966.
The author is with the Institute for Exploratory Research, U. S. Army Electronics Command, Fort Monmouth, N. J.

[1] According to Steier [8] the effects of random lens displacements in the axial direction are coupled to those of the lateral lens displacements and, if the number of lenses of the guide is extremely large, the coupled effects can greatly influence the transmission properties of the beam waveguide. In practical cases, however, as, for example, in the case of a confocal guide with less than 10^4 lenses and with a relative standard displacement of the lenses in the axial direction $\bar{s}_d = \Delta d/d = 10^{-3}$, the axial lens displacements will influence a transmitted wave beam only insignificantly.

the integral equation can be solved in closed form: the eigenfunctions are Gauss-Hermite functions, whose (in general complex) arguments depend on the mean square displacement of the lenses and whose eigenvalues are algebraic functions of this quantity.

Statement and Mathematical Formulation of the Problem

Consider a beam waveguide of equally spaced identical lenses whose focal length is half the lens spacing (see Fig. 1). The axis of the waveguide coincides with the z-axis of a cartesian coordinate system x, y, z. The apertures of the lenses are assumed to be rectangular in shape with the dimensions 2a and 2b in the x- and y-directions. The lens centers are displaced from their proper position on the waveguide axis in a direction perpendicular to this axis. The x- and y-components of the displacements are denoted by s_ν and t_ν, respectively. The phase transformation of the νth lens is then given by

$$\exp\left\{-j\psi_0 + \frac{k}{d}\left[(x - s_\nu)^2 + (y - t_\nu)^2\right]\right\} \tag{1}$$

where ψ_0 is the phase retardation in the lens axis and k the wavenumber. The transformation takes place within the range $-a+s_\nu \leq x \leq +a+s_\nu$, $-b+t_\nu \leq y \leq +b+t_\nu$ outside this range the field is assumed to be absorbed by an opaque screen. We assume that the displacements s_ν, t_ν have a Gaussian probability distribution

$$w(s_\nu) = \frac{e^{-1/2(s_\nu/\bar{s})^2}}{\sqrt{2\pi}\bar{s}}, \qquad w(t_\nu) = \frac{e^{-1/2(t_\nu/\bar{s})^2}}{\sqrt{2\pi}\bar{s}} \tag{2}$$

where the mean square displacement $\bar{s}$ is the same for every ν and for both cross-sectional directions. The sequence of lenses forming the guide is then a structure periodic in the statistical sense.

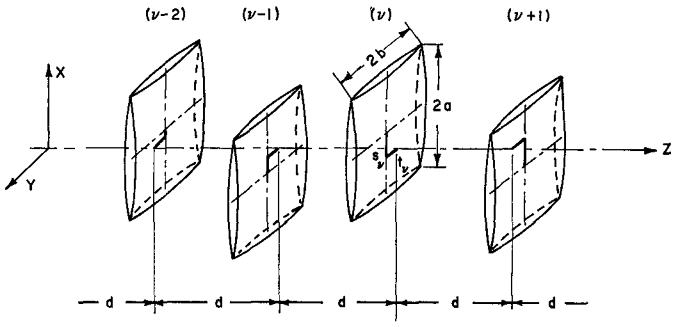

Fig. 1. A section of a misaligned beam waveguide with rectangular lenses.

If the distribution of the tangential electric or magnetic field components of a time harmonic wavebeam is known in the plane of the first lens $z = z_1$ the field in any plane $z > z_1$ can be determined. Assuming at first deterministic displacements s_ν, t_ν the field in the plane z = constant will depend on the displacements of the lenses through which the beam has passed before reaching this plane. Since the displacements are at random, one can define an "expected beam" which is determined by the average of all the possible field distributions.

If any given beam has passed a large number of randomly misaligned lenses, it is reasonable to assume that the expected beam becomes independent of the number of preceding lenses. If this is the case, the expected beam before and after each iteration should be the same, apart from a decrease in amplitude that may be characterized by an "expected" attenuation factor.

We denote the x- or y-component of any one of the possible field distributions *in front* of the νth lens by $E(x_\nu, y_\nu, z_\nu)$. This distribution depends, of course, on the displacements of the preceding lenses. The field E' *behind* the lens is then

$$E'(x_\nu, y_\nu, z_\nu) = \begin{cases} E(x_\nu, y_\nu, z_\nu) \exp\left\{-j\psi_0 + j\dfrac{k}{d}\left[(x_\nu - s_\nu)^2 + (y_\nu - t_\nu)^2\right]\right\} & \text{inside} \\ 0 & \text{outside} \end{cases} \left\{\begin{matrix} -a + s_\nu \leq x \leq +a + s_\nu \\ -b + t_\nu \leq y \leq +b + t_\nu \end{matrix}\right. . \tag{3}$$

From this distribution the field in front of the $(\nu+1)$st lens is derived with the help of Green's function for the plane screen. Using the well-known Fresnel-Kirchhoff approximation of this function we obtain

$$E(x_{\nu+1}, y_{\nu+1}, z_{\nu+1}) = \frac{j}{2\pi}\frac{k}{d}e^{-jkd}\int_{-a+s_\nu}^{+a+s_\nu}\int_{-b+t_\nu}^{+b+t_\nu} E'(x_\nu, y_\nu, z_\nu)\exp\left\{-j\frac{k}{2d}\left[(x_{\nu+1} - x_\nu)^2 + (y_{\nu+1} - y_\nu)^2\right]\right\} dy_\nu dx_\nu$$
$$-\infty \leq x_{\nu+1}, y_{\nu+1} \leq +\infty. \tag{4}$$

Inserting (3) into (4) yields the following relation between the fields at the input planes of two successive lenses

$$E(x_{\nu+1}, y_{\nu+1}, z_{\nu+1}) = \frac{j}{2\pi}\frac{k}{d}e^{-j(kd+\psi_0)}\int_{-a+s_\nu}^{+a+s_\nu}\int_{-b+t_\nu}^{+b+t_\nu} E(x_\nu, y_\nu, z_\nu)$$
$$\cdot\exp\left\{j\frac{k}{d}\left[(x_\nu - s_\nu)^2 + (y_\nu - t_\nu)^2\right] - j\frac{k}{2d}\left[(x_{\nu+1} - x_\nu)^2 + (y_{\nu+1} - y_\nu)^2\right]\right\} dy_\nu dx_\nu. \tag{5}$$

Iterating (5) one can express the field at the $(n+1)$th lens by the field at the first lens

$$E(x_{n+1}, y_{n+1}, z_{n+1}) = \left(\frac{j}{2\pi}\frac{k}{d}\right)^n e^{-jn(kd+\psi_0)} \int_{-a+s_n}^{+a+s_n}\int_{-b+t_n}^{+b+t_n} \cdots \int_{-a+s_1}^{+a+s_1}\int_{-b+t_1}^{+b+t_1} E(x_1, y_1, z_1) \cdot \exp\left\{j\frac{k}{d}\sum_{\nu=1}^{n}\left[(x_\nu - s_\nu)^2 + (y_\nu - t_\nu)^2\right]\right\} \exp\left\{-j\frac{k}{2d}\sum_{\nu=1}^{n}\left[(x_{\nu+1} - x_\nu)^2 + (y_{\nu+1} - y_\nu)^2\right]\right\} \cdot dy_1 dx_1 \cdots dy_n dx_n. \quad (6)$$

The field at the input plane of the $(n+1)$th lens depends on the displacements of all the lenses up to the nth lens. The *expected* field at the $(n+1)$th lens is given by

$$\overline{E}(x_{n+1}, y_{n+1}, z_{n+1}) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} E(x_{n+1}, y_{n+1}, z_{n+1}) w(s_1) w(t_1) \cdots w(s_n) w(t_n) dt_1 ds_1 \cdots dt_n ds_n \quad (7)$$

where the w's are the probability distribution functions, (2).

We now ask the question, is there an incident beam whose *expected* field distribution at lens $n+1$ is (apart from an amplitude factor p) the same as the *actual* distribution at lens 1, or in mathematical formulation, can the equation

$$\overline{E}(x, y, z_{n+1}) = pE(x, y, z_1) \qquad -\infty \leq x, y \leq +\infty \quad (8)$$

be satisfied? With (6) and (7), (8) leads to a homogeneous integral equation of the second kind for the initial field distribution $E(x, y, z_1)$ at the input plane of the first lens[2]

$$pE(x_{n+1}, y_{n+1}, z_1) = \int_{s_n, t_n \cdots s_1,}^{+\infty} \cdots \int_{t_1=-\infty}^{+\infty} \int_{x_n=-a+s_n}^{+a+s_n}\int_{y_n=-b+t_n}^{+b+t_n} \cdots \int_{x_1=-a+s_1}^{+a+s_1}\int_{y_1=-b+t}^{+b+t_1} E(x_1, y_1, z_1) \cdot \prod_{\nu=1}^{n} K(x_{\nu+1}, x_\nu; y_{\nu+1}, y_\nu; s_\nu, t_\nu) dy_1 dx_1 \cdots dy_n dx_n dt_1 ds_1 \cdots dt_n ds_n \quad (9)$$

with

$$K(x_{\nu+1}, x_\nu; y_{\nu+1}, y_\nu; s_\nu, t_\nu) = j\frac{k}{4\pi^2\bar{s}^2 d} e^{-j(kd+\psi_0)} \cdot \exp\left\{j\frac{k}{d}\left[(x_\nu - s_\nu)^2 + (y_\nu - t_\nu)^2\right] - j\frac{k}{2d}\left[(x_{\nu+1} - x_\nu)^2 + (y_{\nu+1} - y_\nu)^2\right] - \frac{1}{2}\frac{s_\nu^2 + t_\nu^2}{\bar{s}^2}\right\}. \quad (10)$$

We interchange the order of integration in that we form pairs of integrals over s_ν, x_ν and over t_ν, y_ν. We then introduce the transformation

$$\int_{s_\nu=-\infty}^{+\infty}\int_{x_\nu=-a+s_\nu}^{+a+s_\nu} \cdots dx_\nu ds_\nu = \int_{x_\nu=-\infty}^{+\infty}\int_{s_\nu=-a+x_\nu}^{+a+x_\nu} \cdots ds_\nu dx_\nu \quad (11)$$

and an analogous transformation for the integration over y_ν, t_ν. This transformation is evident from Fig. 2 where the range of integration is shown in an x_ν, s_ν-plane.

Thus the integral equation (9) can be written

$$pE(x_{n+1}, y_{n+1}, z_1) = \int_{x_n, y_n=-\infty}^{+\infty} \cdots \int_{x_1, y_1,=-\infty}^{+\infty} E(x_1, y_1, z_1) \cdot \prod_{\nu=1}^{n} \overline{K}(x_{\nu+1}, x_\nu; y_{\nu+1}, y_\nu; \bar{s}) \cdot dy_1 dx_1 \cdots dy_n dx_n \quad (12)$$

[2] On the left-hand side of (9) x and y have been replaced by x_{n+1} and y_{n+1} as the x- and y-coordinates which remain after performing the integrations on the right-hand side have these subscripts.

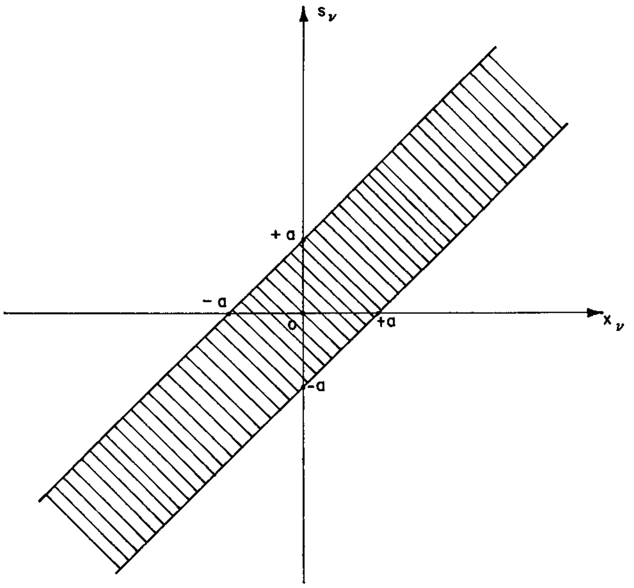

Fig. 2. Geometrical illustration of (11): range of integration in the x_ν, s_ν-plane.

where

$$\overline{K}(x_{\nu+1}, x_\nu; y_{\nu+1}, y_\nu; \bar{s}) = \int_{s_\nu=-a+x_\nu}^{+a+x_\nu} \int_{t_\nu=-b+y_\nu}^{+b+y_\nu} K(x_{\nu+1}, x_\nu; y_{\nu+1}, y_\nu; s_\nu, t_\nu) dt_\nu ds_\nu. \quad (13)$$

The kernel of integral equation (12)

$$\int_{x_n, y_n=-\infty}^{+\infty} \cdots \int_{x_2, y_2=-\infty}^{+\infty} \prod_{\nu=1}^{n} \overline{K}(x_{\nu+1}, x_\nu; y_{\nu+1}, y_\nu; \bar{s}) \cdot dy_2 dx_2 \cdots dy_n dx_n \quad (14)$$

is the $(n-1)$ times iterated kernel K. To solve this equation, therefore, we need only to find the system of eigensolutions of the simpler equation with the noniterated kernel[3]

$$\sqrt[n]{p} E(x_2, y_2, z_1) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E(x_1, y_1, z_1) \overline{K}(x_2, x_1; y_2, y_1; \bar{s}) dy_1 dx_1$$
$$-\infty \leq x_2, y_2 \leq +\infty. \quad (15)$$

This means physically, if the expected field distribution of a wavebeam at the $(n+1)$th lens is equal to the initial field distribution at the first lens, then also the expected field distributions at the second lens, at the third lens etc., are equal to the initial distribution and the wavebeam is reiterative in the statistical sense with the spacing of the lenses. Wavebeams of this property, therefore, will be called the modes of the randomly misaligned waveguide. The expected distribution of these statistical mode functions at the lens planes is determined obviously by the eigenfunctions of the kernel $\overline{K}$ and the expected attenuation factors per iteration are determined by the corresponding eigenvalues. Introducing normalized variables

[3] The eigenvalue $p=0$ which would not permit this statement will not occur for physical reasons.

$$\left.\begin{matrix} \xi, \eta \\ \alpha, \beta \\ \sigma, \tau \end{matrix}\right\} = \sqrt{\frac{k}{d}} \left\{\begin{matrix} x, y \\ a, b \\ s, t \end{matrix}\right. \quad (16)$$

and substituting

$$F(\xi, \eta) = E(x, y, z_1) \exp\left[j\frac{k}{2d}(x^2 + y^2)\right],$$

$$q = \sqrt[n]{p} \exp\left[j\left(\psi_0 + kd - \frac{\pi}{2}\right)\right]$$

$$T(\xi_2, \xi_1; \eta_2, \eta_1) = \overline{K}(x_2, x_1; y_2, y_1; \bar{s}) \exp\left[j\left(\psi_0 + kd - \frac{\pi}{2}\right)\right] \cdot \exp\left[j\frac{k}{2d}\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}\right] \quad (17)$$

integral equation (15) becomes

$$qF(\xi_2, \eta_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\xi_1, \eta_1) T(\xi_2, \xi_1; \eta_2, \eta_1) d\eta_1 d\xi_1$$
$$-\infty \leq \xi_2, \eta_2 \leq +\infty. \quad (18)$$

The kernel T is obtained with (10), (13), and (17)

$$T(\xi_2, \xi_1; \eta_2, \eta_1) = \frac{1}{4\pi^2\sigma^2} \exp[j(\xi_2\xi_1 + \eta_2\eta_1)] \cdot \int_{\sigma_1=-\alpha+\xi_1}^{+\alpha+\xi_1} \int_{\tau_1=-\beta+\eta_1}^{+\beta+\eta_1} \exp[-2j(\xi_1\sigma_1 + \eta_1\tau_1)] \cdot \exp\left[-\left(\frac{1}{2\sigma^2} - j\right)(\sigma_1^2 + \tau_1^2)\right] d\tau_1 d\sigma_1$$

$$= \frac{1}{8\pi\gamma^2} \exp[j(\xi_2\xi_1 + \eta_2\eta_1)] \cdot \exp\left[-2\left(\frac{\sigma}{\gamma}\right)^2 (\xi_1^2 + \eta_1^2)\right] \cdot \left\{\Phi\left(\frac{\gamma\alpha + \xi_1/\gamma}{\sqrt{2}\sigma}\right) - \Phi\left(\frac{-\gamma\alpha + \xi_1/\gamma}{\sqrt{2}\sigma}\right)\right\} \cdot \left\{\Phi\left(\frac{\gamma\beta + \eta_1/\gamma}{\sqrt{2}\sigma}\right) - \Phi\left(\frac{-\gamma\beta + \eta_1/\gamma}{\sqrt{2}\sigma}\right)\right\} \quad (19)$$

where Φ is the error integral

$$\Phi(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-v^2} dv \quad \text{and} \quad \gamma = (1 - 2j\sigma^2)^{1/2} \quad (20)$$

$\sigma = \sqrt{(k/d)}\bar{s}$ is the normalized mean square displacement of the lenses. Since the kernel T is not Hermitian, we do not have a mathematical theorem to prove the existence of eigensolutions of (18). To be able to proceed we shall assume, however, in the following that this equation has indeed solutions. In particular, if the mean square displacement is small, the eigenfunctions will not deviate essentially from the characteristic functions for $\sigma=0$, i.e., from the mode functions of the perfectly aligned guide which even form a complete system.

The Orthogonality Relation of the Statistical Mode Functions

The eigenfunctions of (18) which, in general, are complex satisfy the orthogonality relation

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F_\mu(\xi,\eta)F_m(\xi,\eta)W(\xi,\eta)d\eta d\xi = \begin{cases} A_m & \mu = m \\ 0 & \mu \neq m. \end{cases} \quad (21)$$

F_μ and F_m are any two eigenfunctions; if they are properly normalized the constant A_m is unity.

In contrast to the mode functions for the aligned guide the orthogonality relation for the statistical mode functions requires a weight function

$$W(\xi,\eta) = \frac{1}{4\gamma^2}\exp\left[-2\left(\frac{\sigma}{\gamma}\right)^2(\xi^2+\eta^2)\right] \cdot \left\{\Phi\left(\frac{\gamma\alpha+\xi/\gamma}{\sqrt{2}\,\sigma}\right) - \Phi\left(\frac{-\gamma\alpha+\xi/\gamma}{\sqrt{2}\,\sigma}\right)\right\} \cdot \left\{\Phi\left(\frac{\gamma\beta+\eta/\gamma}{\sqrt{2}\,\sigma}\right) - \Phi\left(\frac{-\gamma\beta+\eta/\gamma}{\sqrt{2}\,\sigma}\right)\right\}. \quad (22)$$

The orthogonality relation (21) follows from the fact that with the substitution

$$\hat{F}(\xi,\eta) = F(\xi,\eta\sqrt{W(\xi,\eta)} \quad (23)$$

(18) transforms into an integral equation with the symmetric kernel

$$\hat{T}(\xi_2,\xi_1;\eta_2,\eta_1) = \frac{1}{2\pi}e^{j(\xi_2\xi_1+\eta_2\eta_1)}\sqrt{W(\xi_2,\eta_2)W(\xi_1,\eta_1)}. \quad (24)$$

But the eigenfunctions of a symmetric kernel are orthogonal to one another with the weight function 1.

Expected Attenuation and Expected Power Loss of the Statistical Mode Functions

Consider the transmission system shown in Fig. 3. A transmitter excites in the input plane of the first lens the initial field distribution of a statistically reiterative wavebeam. The wavebeam is then passed through a sequence of n (misaligned) lenses, and is received by a calibrated receiver, placed at a distance d behind the nth lens [i.e., at the input plane of the $(n+1)$th lens]. The voltage V_{n+1}, measured by the receiver, is compared to the reference voltage V_1 which is measured by the same receiver when placed immediately at the input plane of the first lens.

The expected value of the voltage ratio V_{n+1}/V_1 obviously is the expected amplitude attenuation factor of the wavebeam over the path length of the sequence of n lenses and

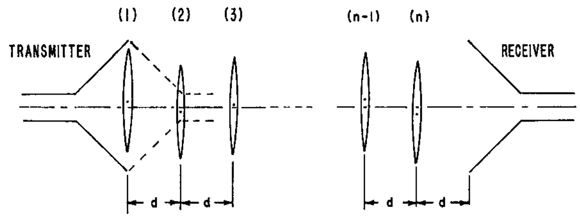

Fig. 3. Misaligned beam waveguide with transmitter and receiver.

the expected value of the absolute square voltage ratio $V_{n+1}\cdot V_{n+1}^*/V_1V_1^*$ is the expected power attenuation factor over the same path length.

The voltage measured by the receiver is

$$V = \int_S\int U(\xi,\eta)\psi(\xi,\eta)d\eta d\xi \quad (25)$$

where $U(\xi,\eta)$ is the distribution of any given incident field in the aperture plane S of the receiver and $\psi(\xi,\eta)$ is a characteristic function of the receiver. Since we consider a statistical beam mode, whose actual field distribution in the input plane of the first lens is given by an eigenfunction of integral equation (18), $U_1(\xi,\eta)=F(\xi,\eta)$, and whose expected field distribution in the input plane of the $(n+1)$th lens is given by the same function, $\overline{U}_{n+1}(\xi,\eta)=q^nF(\xi,\eta)$, where q is the corresponding eigenvalue, we obtain with (25):

$$\overline{V}_{n+1} = q^nV_1 = q^n\int_S\int F(\xi,\eta)\psi(\xi,\eta)d\eta d\xi. \quad (26)$$

The expected voltage ratio $\overline{V}_{n+1}/V_1$, therefore, is equal to q^n and the eigenvalue q is the expected amplitude attenuation factor per iteration.

According to a well-known theorem of probability theory, the expected value of the absolute square voltage $V_{n+1}\cdot V_{n+1}^*$ can be written

$$\overline{V_{n+1}V_{n+1}^*} = \overline{V}_{n+1}\overline{V}_{n+1}^* + \overline{\Delta V_{n+1}\Delta V_{n+1}^*} \quad (27)$$

where $\overline{\Delta V_{n+1}\Delta V_{n+1}^*}$ is the variance of V_{n+1}. Since the variance is a non-negative quantity (27) can be written as an inequality $\overline{V_{n+1}V_{n+1}^*}\geq\overline{V}_{n+1}\overline{V}_{n+1}^*$, and if we divide by $V_1\cdot V_1^*$ we have on the left-hand side of this inequality the expected power attenuation factor of the wavebeam over a distance of n lens sections and on the right-hand side the absolute square of the expected amplitude attenuation factor over the same distance, which according to (26) is equal to $(qq^*)^n$:

$$\frac{\overline{V_{n+1}V_{n+1}^*}}{V_1V_1^*} \geq (qq^*)^n. \quad (28)$$

Since this inequality holds for any number of lenses n, the absolute square eigenvalues determine a lower bound for the expected power attenuation factor per iteration and correspondingly an upper bound for the expected power loss of the statistically reiterative wavebeams.

Discussion of the Kernel of the Statistical Mode Equation

The kernel (19) is s product of two kernels, the first one depending only on ξ_2, ξ_1, α the other one depending only on η_2, η_1, β. Apart from this difference the two kernels are identical:

$$T(\xi_2, \xi_1; \eta_2, \eta_1; \alpha, \beta) = M(\xi_2, \xi_1; \alpha)M(\eta_2, \eta_1; \beta). \quad (29)$$

Hence, if the kernel $M(\xi_2, \xi_1, \alpha)$ has the eigenfunctions $G_m(\xi, \alpha)$ and the eigenvalues $\lambda_m(\alpha)$ any product function

$$F(\xi, \eta; \alpha, \beta) = G_m(\xi; \alpha)G\mu(\eta; \beta) \quad (30)$$

is an eigenfunction of T and the corresponding eigenvalue is

$$q(\alpha, \beta) = \lambda_m(\alpha)\lambda\mu(\beta). \quad (31)$$

With this in mind, we shall discuss only the kernel M which applies physically to the two-dimensional problem of a guide with cylindrical lenses.

The kernel M is again a product of three terms. The first term

$$M_1(\xi_2, \xi_1) = \frac{1}{\sqrt{2\pi}} e^{j\xi_2\xi_1} \quad (32)$$

is the Fourier kernel of the perfectly aligned beam waveguide. The second term

$$M_2(\xi_1) = \frac{1}{\gamma}\exp\left[-2\left(\frac{\sigma}{\gamma}\right)^2 \xi_1^2\right], \gamma = (1 - 2j\sigma^2)^{1/2} \quad (33)$$

is caused by the lateral displacement of the *phase transformation:* if only the apertures were misaligned and not the axis of the lenses, this term would reduce to unity. The third term

$$M_3(\xi_1; \alpha) = \frac{1}{2}\left\{\Phi\left(\frac{\gamma\alpha + \xi_1/\gamma}{\sqrt{2}\,\sigma}\right) - \Phi\left(\frac{-\gamma\alpha + \xi_1/\gamma}{\sqrt{2}\,\sigma}\right)\right\} \quad (34)$$

takes into account the lateral displacement of the *apertures.* For perfectly aligned apertures, M_3 reduces to a step function of value 1 within the area of the apertures and of value zero outside this area

$$M_3(\xi_1; \alpha) = \begin{cases} 1 & |\xi_1| < \alpha \\ 0 & |\xi_1| > \alpha \end{cases} \quad \text{for } \sigma = 0$$

thus in effect limiting the range of integration in (18) to the area of the apertures. For misaligned apertures ($\sigma > 0$) the discontinuous step from 1 to 0 is smoothed out into a continuous transition which becomes more gradual with increasing σ. Figure 4 shows the distribution of M_3 for $\sigma/\alpha = 1$ percent, 2 percent, and 5 percent, assuming $\gamma = (1 - 2j\sigma^2)^{1/2} \approx 1$

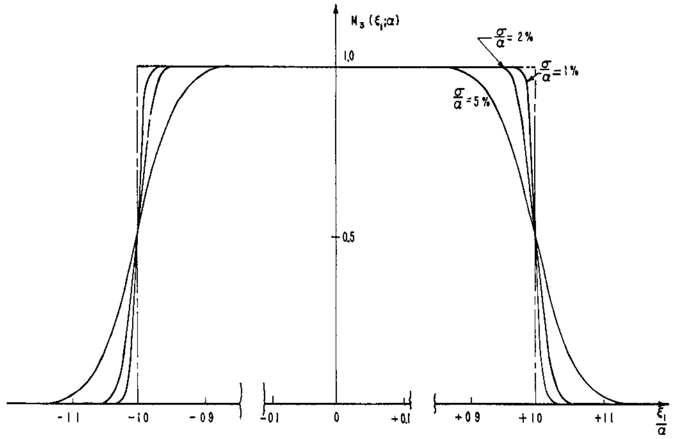

Fig. 4. The term M_3 in the kernel of integral equation (36) as a function of ξ_1/α for various values of the relative mean square displacement σ/α. For $\sigma \to 0$, M_3 approaches a step function as indicated by the dotted line.

The integral equation with the kernel $M = M_1M_2M_3$,

$$\lambda_m G_m(\xi_2) = \int_{-\infty}^{+\infty} G_m(\xi_1)M(\xi_2, \xi_1)d\xi_1, \qquad -\infty \leq \xi_2 \leq +\infty \quad (36)$$

obviously satisfies the requirement that for $\sigma = 0$ it approaches the integral equation of the perfectly aligned beam waveguide with the Fourier kernel (32). The eigenfunctions and eigenvalues of this equation are angular and radial prolate spheroidal wave functions, [11] respectively:

$$G_m(\xi) = S_{0m}\left(\alpha^2, \frac{\xi}{\alpha}\right), \quad \lambda_m = j^m\sqrt{\frac{2}{\pi}}\,\alpha R_{0m}^{(1)}(\alpha^2, 1) \qquad \text{for } \sigma = 0. \quad (37)$$

If $\sigma > 0$ it seems unlikely that the eigensolutions of (36) can still be written as simple expressions in known functions. Therefore, we restrict ourselves to solving this equation in two special cases: in the first case we assume small mean square displacements σ and solve (36) by a perturbation method. In the second case we assume infinitely extended apertures; (36) can then be solved rigorously.

The Slightly Misaligned Beam Waveguide

In practice it will not present a major technological problem to align the lenses in an actual beam waveguide with a tolerance that is small as compared to the radius of the apertures. The case of slight misalignment, therefore, is primarily of interest, from the point of view of applications.

Assuming $\sigma \ll \alpha$ and also $\sigma \ll 1$ we simplify the kernel M of integral equation (36). The term M_1 remains unchanged, since according to (32) this term does not depend on σ. The term M_2, (33), is in a second-order approximation:

$$M_2(\xi_1) = 1 + \sigma^2(j - 2\xi_1^2). \quad (38)$$

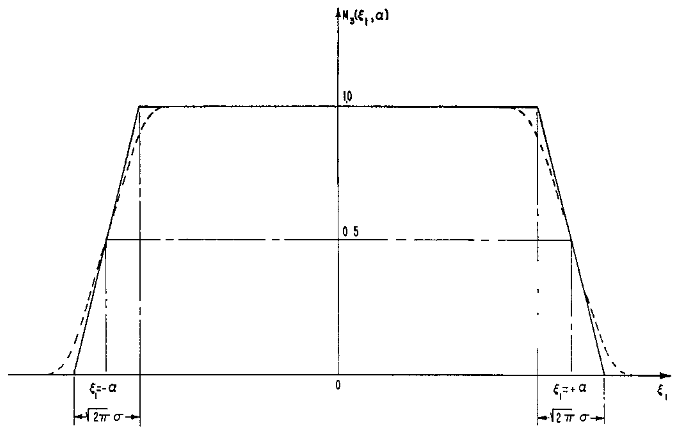

Fig. 5. Approximation of the term $M_3(\xi_1;\alpha)$ in the kernel of integral equation (36) by a trapezoidal function formed by the tangents to M_3 in the points $\xi_1=0,\ \pm\alpha,\ \pm\infty$.

In the term M_3, (34), we approximate the argument of Φ, i.e., $(\gamma\alpha+\xi/\gamma)/\sqrt{2}\sigma$ by $\alpha+\xi/\sqrt{2}\sigma$; $M_3(\xi_1,\alpha)$ then becomes a real function.[4] We approximate this function by a trapezoidal distribution which, as shown in Fig. 5, replaces $M_3(\xi,\alpha)$ by its tangents in the points $\xi_1=0,\ \pm\alpha,\ \pm\infty$:

$$M_3(\xi_1,\alpha)=\begin{cases}1 & 0\le|\xi_1|\le\alpha-\sqrt{\dfrac{\pi}{2}}\,\sigma\\[2mm] \dfrac{1}{2}\left(1+\sqrt{\dfrac{2}{\pi}}\,\dfrac{\alpha-|\xi_1|}{\sigma}\right) & \alpha-\sqrt{\dfrac{\pi}{2}}\,\sigma\le|\xi_1|\le\alpha+\sqrt{\dfrac{\pi}{2}}\,\sigma\\[2mm] 0 & \alpha+\sqrt{\dfrac{\pi}{2}}\,\sigma\le|\xi_1|\le\infty\end{cases}\qquad(39)$$

With the approximations (38) and (39) integral equation (36) becomes

$$\begin{aligned}\lambda_m G_m(\xi_2)&=\int_{-\alpha}^{+\alpha}\theta_m(\xi_2,\xi_1)d\xi_1\\&+\frac{1}{2}\int_{-\alpha-\sqrt{(\pi/2)}\sigma}^{-\alpha}\theta_m(\xi_2,\xi_1)\left(1+\sqrt{\frac{2}{\pi}}\,\frac{\alpha+\xi_1}{\sigma}\right)d\xi_1-\frac{1}{2}\int_{-\alpha}^{-\alpha+\sqrt{(\pi/2)}\sigma}\theta_m(\xi_2,\xi_1)\left(1-\sqrt{\frac{2}{\pi}}\,\frac{\alpha+\xi_1}{\sigma}\right)d\xi_1\\&-\frac{1}{2}\int_{+\alpha-\sqrt{(\pi/2)}\sigma}^{+\alpha}\theta_m(\xi_2,\xi_1)\left(1-\sqrt{\frac{2}{\pi}}\,\frac{\alpha-\xi_1}{\sigma}\right)d\xi_1+\frac{1}{2}\int_{+\alpha}^{+\alpha+\sqrt{(\pi/2)}\sigma}\theta_m(\xi_2,\xi_1)\left(1+\sqrt{\frac{2}{\pi}}\,\frac{\alpha-\xi_1}{\sigma}\right)d\xi_1\end{aligned}\qquad(40)$$

where $\theta(\xi_2,\xi_1)$ stands short for

$$M_1(\xi_2,\xi_1)M_2(\xi_1)G_m(\xi_1)\approx\frac{1}{\sqrt{2\pi}}e^{j\xi_2\xi_1}[1+\sigma^2(j-2\xi_1^2)]G_m(\xi_1).\qquad(41)$$

The range of the first integral on the right-hand side of (40) has the width 2α, while the width of the ranges of the remaining four integrals is $\sqrt{(\pi/2)}\sigma\ll1$. In these integrals, we expand the function $\theta_m(\xi_2,\xi_1)$ into a power series of ξ_1 (at $\xi_1=+\alpha$ or $\xi_1=-\alpha$, respectively). The integrations can then be performed leading to power series in σ. Considering terms up to the second order we obtain the approximate integral equation

$$\begin{aligned}\lambda_m G_m(\xi_2)&=\frac{1}{\sqrt{2\pi}}\int_{-\alpha}^{+\alpha}G_m(\xi_1)e^{j\xi_2\xi_1}d\xi_1\\&+\frac{\sigma^2}{\sqrt{2\pi}}\int_{-\alpha}^{+\alpha}G_m(\xi_1)e^{j\xi_2\xi_1}(j-2\xi_1^2)d\xi_1\\&+\sigma^2\frac{\sqrt{2\pi}}{24}\{[G_m'(\alpha)+j\xi_2G_m(\alpha)]e^{j\xi_2\alpha}\\&-[G_m'(-\alpha)\\&j+\xi_2G_m(-\alpha)]e^{-j\xi_2\alpha}\}\quad -\alpha\le\xi_2\le+\alpha.\end{aligned}\qquad(42)$$

The first σ^2-term on the right-hand side can be attributed to the displacement of the phase transformation and the second σ^2-term to the displacement of the apertures.

Since σ is assumed to be small the eigenfunctions and eigenvalues of (42) will not substantially deviate from the corresponding quantities (37) for $\sigma=0$; hence (42) can be solved by a perturbation method. As this method follows a standard procedure we shall omit the intermediate steps and only state the result:

$$\begin{aligned}G_m(\xi)&=S_{0m}\left(\alpha^2,\frac{\xi}{\alpha}\right)+\sigma^2\sum_{\mu=0}^{\infty}c_{m\mu}S_{0\mu}\left(\alpha^2,\frac{\xi}{\alpha}\right)\\ \lambda_m&=\lambda_{0m}+\sigma^2\kappa_m\quad\text{with}\\ \lambda_{0m}&=j^m\sqrt{\frac{2}{\pi}}\,\alpha\,R_{0m}^{(1)}(\alpha^2,1)\end{aligned}\qquad(43)$$

[4] M_3 does not have a power series expansion in σ near $\sigma=0$. In simplifying M_3 we therefore have to take an approach different from the second-order approximation we shall otherwise use in this section.

where

$$
\left.
\begin{aligned}
c_{m\mu} &= c_{m\mu}{}^{(1)} + c_{m\mu}{}^{(2)}, \quad \kappa_m = \kappa_m{}^{(1)} + \kappa_m{}^{(2)} \\
c_{m\mu}{}^{(1)} &= \frac{1}{1-\dfrac{\lambda_{0m}}{\lambda_{0\mu}}}\,\frac{2}{\Gamma_\mu}\int_{-\alpha}^{+\alpha} S_{0m}\left(\alpha^2, \frac{\xi}{\alpha}\right) S_{0\mu}\left(\alpha^2, \frac{\xi}{\alpha}\right)\xi^2 d\xi \\
c_{m\mu}{}^{(2)} &= -\frac{1}{1-\dfrac{\lambda_{0m}}{\lambda_{0\mu}}}\,\frac{\pi}{6\Gamma_\mu}\left[S_{0m}(\alpha^2,1)S_{0\mu}'(\alpha^2,1) + S_{0\mu}(\alpha^2,1)S_{0m}'(\alpha^2,1)\right],
\end{aligned}
\right\}\ \text{for } m+\mu\text{: even and } m \neq \mu
$$

$$c_{m\mu}{}^{(1)} = c_{m\mu}{}^{(2)} = 0 \qquad \text{for } m+\mu\text{: odd and } m = \mu$$

$$\kappa_m{}^{(1)} = \lambda_{0m}\left\{j - \frac{2}{\Gamma_m}\int_{-\alpha}^{+\alpha} S_{0m}{}^2\left(\alpha^2, \frac{\xi}{\alpha}\right)\xi^2 d\xi\right\}$$

$$\kappa_m{}^{(2)} = \lambda_{0m}\frac{\pi}{3\Gamma_m} S_{0m}(\alpha^2, 1) S_{0m}'(\alpha^2, 1)$$

$$\text{with } S_{0m}'\left(\alpha^2, \frac{\xi}{\alpha}\right) = \frac{\partial}{\partial \xi} S_{0m}\left(\alpha^2, \frac{\xi}{\alpha}\right), \quad \Gamma_m = \int_{-\alpha}^{+\alpha} S_{0m}{}^2\left(\alpha^2, \frac{\xi}{\alpha}\right) d\xi. \tag{44}$$

Each coefficient $c_{m\mu}$, κ_m consists of two terms: the first term pertains to the displacement of the phase transformation, the second term to the displacement of the apertures. For sufficiently large apertures ($\alpha > 2 \cdots 3$) the functions $S_{0m}(\alpha^2, \xi/\alpha)$, in particular those of small order m, decrease to very small values at $\xi = \alpha$. Hence, in this case the terms $c_{m\mu}{}^{(2)}$, $\kappa_m{}^{(2)}$ become small as compared to the terms $c_{m\mu}{}^{(1)}$, $\kappa_m{}^{(1)}$. An estimate is obtained by approximating the functions S_{0m} by their asymptotic Gauss-Hermite representations [3], [11]

$$S_{0m}\left(\alpha^2, \frac{\xi}{\alpha}\right) \to \text{const. } He_m(\sqrt{2}\,\xi)e^{-(1/2)\xi} \qquad \text{for } \alpha \to \infty. \tag{45}$$

In the case of the coefficients κ of the two lowest-order eigenvalues λ_0 and λ_1 the terms with superscripts (1) and (2) become

$$
\begin{aligned}
&\text{for } m = 0 \quad
\begin{cases}
\dfrac{\kappa_0{}^{(1)}}{\lambda_{00}} = j - 1 + \dfrac{2}{\sqrt{\pi}}\alpha e^{-\alpha^2} \\[2ex]
\dfrac{\kappa_0{}^{(2)}}{\lambda_{00}} = -\dfrac{\sqrt{\pi}}{3}\alpha e^{-\alpha^2}
\end{cases} \\
&\hspace{20em} (\alpha > 2 \cdots 3) \\
&\text{for } m = 1 \quad
\begin{cases}
\dfrac{\kappa_1{}^{(1)}}{\lambda_{01}} = j - 3 + \dfrac{4}{\sqrt{\pi}}\alpha^3 e^{-\alpha} \\[2ex]
\dfrac{\kappa_1{}^{(2)}}{\lambda_{01}} = -\dfrac{2\sqrt{\pi}}{3}\alpha(\alpha^2 - 1)e^{-\alpha^2}.
\end{cases}
\end{aligned}
\tag{46}
$$

Since the eigenvalues determine the expected attenuation of the statistical mode functions it can be stated that the effect of misalignment on the attenuation is caused essentially by the displacement of the phase transformation, while the displacement of the apertures amounts only to a correction of the former effect.

The absolute square eigenvalues $\lambda_0\lambda_0{}^*$ and $\lambda_1\lambda_1{}^*$ have been calculated with (43) and (46) for $\sigma = 0.01$, 0.02, and 0.05. In Fig. 6, the quantities $1 - \lambda_0\lambda_0{}^*$ and $1 - \lambda_1\lambda_1{}^*$ (which are upper bounds of the expected power loss per iteration for the lowest-order symmetric and antisymmetric statistical mode functions, respectively) have been plotted as functions of α. For small α-values the curves follow closely the diffraction loss curves of the perfectly aligned waveguide as indicated by the dotted lines. With increasing α the diffraction loss decreases rapidly and the curves approach a limiting value determined by the mean square displacement σ of the lenses.

The Misaligned Beam Waveguide with Infinitely Extended Apertures

We now consider a beam waveguide with infinitely extended lenses, in other words, we assume that the phase transformation (1) is performed over the entire planes of phase correction. In this case we do not require σ to be a small quantity.

A wavebeam transmitted in this type of idealized waveguide will not suffer any loss of energy. However, due to the misalignment of the lenses the energy distribution over the beam cross section will spread out as the wavebeam is passed from lens to lens. Hence, a receiver, matched to a certain field distribution, will only recover part of the energy launched by the transmitter and an amplitude and power attenuation will be measured.

For $\alpha \to \infty$ the terms M_1 and M_2 in the kernel of integral equation (36) remain unchanged, while

$$M_3(\xi_1; \alpha)_{\alpha \to \infty} \to 1, \qquad -\infty \leq \xi_1 \leq +\infty. \tag{47}$$

Hence (36) reads

$$\lambda_m G_m(\xi_2) = \frac{1}{\sqrt{2\pi}\,\gamma}\int_{-\infty}^{+\infty} G_m(\xi_1) \cdot \exp\left[j\xi_2\xi_1 - 2\left(\frac{\sigma}{\gamma}\right)^2 \xi_1^2\right] d\xi_1 \qquad -\infty \le \xi_2 \le +\infty. \tag{48}$$

The eigenfunctions of this equation are—as in the case $\sigma=0$—Gauss-Hermite functions, but their arguments are complex [12]

$$G_m(\xi) = He_m(u\xi)\exp\left[-\left\{\frac{1}{4}u^2 - \left(\frac{\sigma}{\gamma}\right)^2\right\}\xi^2\right] \qquad m = 0, 1, 2 \cdots$$

$$\text{with } u = \sqrt{2}\left[1 + 4\left(\frac{\sigma}{\gamma}\right)^4\right]^{1/4}, \qquad 0 \le \arg u \le \frac{\pi}{4}$$

$$\gamma = (1 - 2j\sigma^2)^{1/2}. \tag{49}$$

These functions form in the range $-\infty \le \xi \le +\infty$ an orthogonal system with the weight function $\exp[-2(\sigma/\gamma)^2\xi^2]$

$$\int_{-\infty}^{+\infty} G_m(\xi)G_\mu(\xi)\exp\left[-2\left(\frac{\sigma}{\gamma}\right)^2\xi^2\right]d\xi = \begin{cases}\dfrac{\sqrt{2\pi}\,m!}{u} & \text{for } \mu = m\\ 0 & \text{for } \mu \ne m.\end{cases} \tag{50}$$

For $\sigma \ll 1$ the parameter u can be approximated

$$u \approx \sqrt{2}(1 + \sigma^4), \qquad \sigma \ll 1. \tag{51}$$

Hence, even in a fourth-order approximation the argument of the Hermite polynomials becomes real. If this approximation is used the eigenfunctions $G_m(\xi)$be come real function apart from a common factor $\exp[(\sigma/\gamma)^2\xi^2]$; in orthogonality relation (50) this factor is compensated by the weight function. As a consequence this relation allows expansion of a given cross-sectional field distribution into a series of the approximated mode functions according to the principle of minimum mean square deviation. Since the Hermite polynomials of real argument form a complete system for the range $-\infty \le \xi \le +\infty$ the mean square deviation tends to zero and the series of Gauss-Hermite functions actually represents the field.

The expected cross-sectional distribution of an arbitrary wavebeam transmitted in a slightly misaligned beam waveguide with infinitely extended lenses can, therefore, be described by the statistical mode functions in the same way as the actual distribution of a wavebeam transmitted in a perfectly aligned beam waveguide is described by the beam modes.

The eigenvalues of (48) are

$$\lambda_m = j^m \frac{\left[2\left\{\frac{1}{4}u^2 - \left(\frac{\sigma}{\gamma}\right)^2\right\}\right]^{m+1/2}}{\gamma} \qquad m = 0, 1, 2, \cdots. \tag{52}$$

The absolute square eigenvalues

$$\lambda_m\lambda_m^* = \frac{[(1 + 16\sigma^4)^{1/2} - 2\sqrt{2}\,\sigma^2\{1 + (1 + 16\sigma^4)^{1/2}\}^{1/2} + 4\sigma^4]^{m+1/2}}{(1 + 4\sigma^4)^{m+1}} \tag{53}$$

have been calculated for $m=0 \cdots 4$. They are plotted in Fig. 7 as functions of σ. Since for any σ the eigenvalue λ_0 is the one with the largest absolute value, the statistical mode function with the parameter $m=0$ suffers the smallest (expected) attenuation and can therefore be called the dominant mode function. The value (52) for λ_0 has been confirmed by an entirely different method: we consider the misaligned guide of Fig. 3 and assume that the transmitter excites the dominant beam mode of the perfectly aligned guide.[5] The lens displacements do not change the Gaussian distribution of the mode as it travels along the guide, but cause the mode axis to follow a zig-zag path which is determined by the laws of geometrical optics [4]. Assuming that the receiver is matched to the centered beam, the expected receiver output voltage $\overline{V}_{n+1}$ can be calculated from the statistical variations of the zig-zag paths. It can be shown that, as the number n of the lenses increases, the voltage ratio $\overline{V}_{n+1}/\overline{V}_n$ (which determines the expected amplitude attenuation per lens section) approaches the eigenvalue λ_0 as given by (52). The actual calculations will not be presented here since they are rather tedious and involved.

Note that (52), for the eigenvalues in a waveguide with infinitely large lenses, is in agreement with (46) for the eigenvalues in a slightly misaligned waveguide. If, in (52) we consider only terms up to the second order in σ and in (46) have α approach infinity, we obtain in both cases

$$\lambda_m = j^m[1 - (2m + 1 - j)\sigma^2]. \tag{54}$$

[5] The lenses in Fig. 3 in this case are assumed extended.

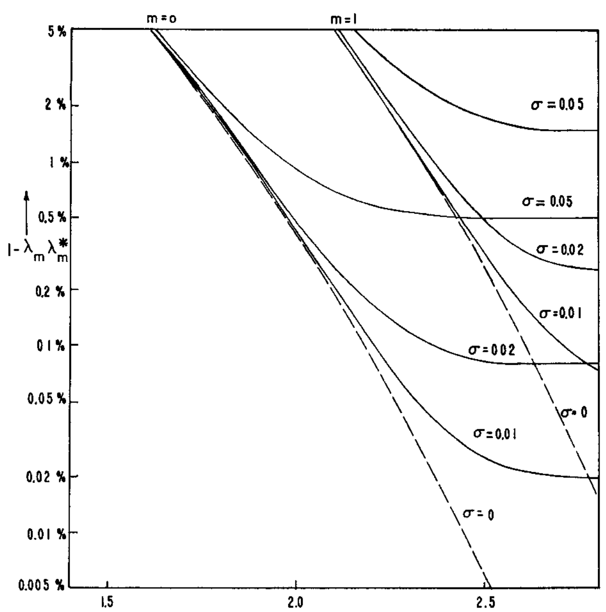

Fig. 6. The absolute squares of the expected attenuation factors λ_0 and λ_1 of the lowest-order symmetric and antisymmetric statistical mode functions in a slightly misaligned beam waveguide ($\sigma \ll 1$). Since $\lambda_0\lambda_0^*$ and $\lambda_1\lambda_1^*$ are very close to 1, the difference of these values to 1 has been plotted in a logarithmic scale.

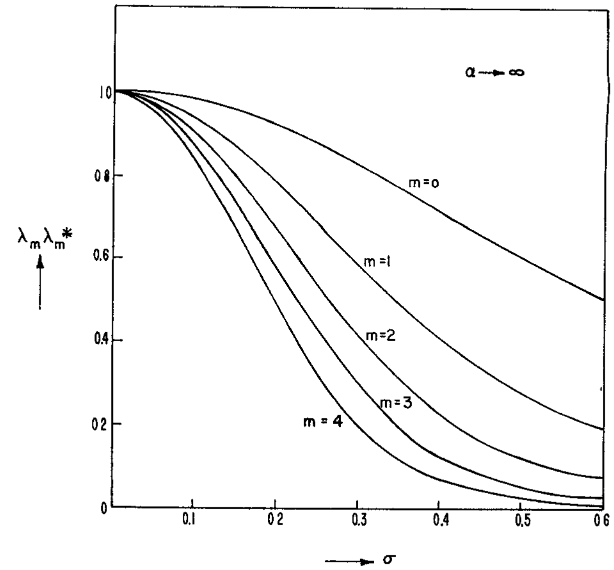

Fig. 7. The absolute squares of the expected attenuation factors of the lower-order statistical mode functions in a misaligned beam waveguide with infinitely extended apertures ($\alpha \rightarrow \infty$).

Acknowledgment

The author thanks Dr. G. Goubau for many helpful discussions on the subject of this paper.

References

[1] G. Goubau and R. Christian, "A new waveguide for millimeter waves," *Proc. Army Science Conf.*, U. S. Military Academy, West Point, N. Y., vol. 1, pp. 291–303, June 1959.

[2] G. Goubau and F. Schwering, "On the guided propagation of electromagnetic wave beams," *IEEE Trans. on Antennas and Propagation*, vol. AP-9, pp. 248–256, May 1961.

[3] F. Schwering, "Reiterative wavebeams of rectangular symmetry," *Arch. Elekt. Übertragung*, vol. 15, pp. 555–564, December 1961.

[4] G. Goubau and R. Christian, "Some aspects of beam waveguides for long distance transmission at optical frequencies," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-12, pp. 212–220, March 1964.

[5] F. Hirano and Y. Fukatsu, "Stability of a light beam in a beam waveguide," *Proc. IEEE*, vol. 52, pp. 1284–1292, November 1964.

[6] D. Marcuse, "Statistical treatment of light-ray propagation in beam wave guides," *Bell Sys. Tech. J.*, vol. 44, pp. 2065–2081, November 1965.

[7] D. W. Berreman, "Growth of oscillations of a ray about the irregularly wavey axis of a lens light guide," *Bell Sys. Tech. J.*, vol. 44, pp. 2117–2133, November 1965.

[8] W. H. Steier, "The statistical effects of random variations on the performance of a beam waveguide," *Bell Sys. Tech. J.*, vol. 45, pp. 451–471, March 1966.

[9] C. Eaglesfield, "Mode conversion loss in a sequential confocal lens system," *Proc. IEE (London)*, vol. 111, pp. 610–615, March 1964.

[10] D. Gloge, "Ein allgemeines verfahren zur berechnung optischer resonatoren und periodischer linsensysteme," *Arch. Elekt. Übertragung*, vol. 19, pp. 13–26, January 1965.

[11] G. D. Boyd and J. P. Gordon, "Confocal multimode resonator for millimeter through optical wavelength masers," *Bell Sys. Tech. J.*, vol. 40, pp. 489–508, March 1961.

[12] F. Oberhettinger, *Tabellen Zur Fourier Transformation.* Berlin: Springer, 1957, p. 45.